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SHORTER COMMUNICATION

ON THE SOLUTION OF GRAETZ TYPE PROBLEMS WITH AXIAL CONDUCTION

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NOMENCLATURE

G,	dimensionless heat generation;
h_i,	heat transfer coefficient, $i = 1, 2$;
H_i,	$= h_iL/k$, Biot number, $i = 1, 2$;
L,	a reference length;
k,	thermal conductivity;
Pe,	$= (\beta \bar{U}_m L/\alpha)$, Péclét number;
r,	radial coordinate;
R,	tube radius;
T,	temperature;
T_0,	inlet temperature;
$\bar{u}_m$,	mean velocity;
x, y,	axial and normal coordinates, respectively;
α,	molecular diffusivity of heat;
β,	coefficient in $u(\eta) = \beta \bar{U}_m \Delta(\eta)$;
$\gamma(j)$,	$= 1 + \dfrac{\varepsilon_H(j)}{\alpha}, j = x, y$;
$\varepsilon_H(j)$,	turbulent diffusivity of heat;
θ,	$= \dfrac{T}{T_0}$, dimensionless temperature;
η,	
ξ,	$= \dfrac{y}{L}$, dimensionless normal coordinate;

INTRODUCTION

NUMEROUS extensions of the original Graetz problem appeared in the literature since its formulation by Graetz in 1885. However, the analysis becomes rather involved mathematically if the axial heat conduction in the fluid [1–18] is to be considered. Therefore, various elaborate mathematical approaches have been tried to solve the problems with axial heat conduction in the fluid. For example, the temperature field was arbitrarily expanded in terms of the Fourier sine series [1]; the nonorthogonal eigenfunctions were expanded in terms of the eigenfunctions of an auxilliary orthogonal system [2, 3]. An expansion similar to that used in references [2, 3] was applied [4] to solve the problems with axial conduction in the upstream and downstream regions of the origin. The Gramm–Schmidt orthonormalization procedure was used to construct orthogonal functions from the nonorthogonal eigenfunctions [11, 12]; the same technique was also applied to solve related problems involving internal energy sources [13, 14]. A two-sided Laplace transform technique was used to represent the temperature field in the upstream and downstream regions of the origin [15–17]; a combination of purely numerical scheme and an asymptotic expansion technique was tried [18].

A scrutiny of all these analytical approaches discussed above reveals that their application is rather involved and their extension to more general situations is a complicated matter.

We now present a straightforward, simple, yet a very general, finite integral transform technique for the solution of Graetz type channel flow heat transfer problems including the axial conduction, convective boundary conditions, and internal energy sources. The analysis is applicable to both laminar and turbulent flow inside conduits. As our objective in this work is to demonstrate the application of the method, we examine only the situation in which the axial heat conduction is considered in the downstream direction. The problems in which the axial conduction is considered in both upstream and downstream directions can also be handled with this approach.

ANALYSIS

We consider a channel flow heat transfer problem with axial conduction given in the dimensionless form as

$$\frac{1}{\eta^s}\frac{\partial}{\partial \eta}\left[\eta^s \gamma(\eta)\frac{\partial \theta}{\partial \eta}\right] + \frac{1}{Pe^2}\frac{\partial}{\partial \xi}\left[\gamma(\xi)\frac{\partial \theta}{\partial \xi}\right] + G(\eta)$$
$$= \Delta(\eta)\frac{\partial \theta(\eta, \xi)}{\partial \xi}, \text{in} \quad \eta_1 < \eta < \eta_2, \quad \xi > 0, \qquad (1a)$$

$$-\frac{\partial \theta}{\partial \eta} + H_1\theta = 0 \qquad \text{at}\, \eta = \eta_1, \qquad \xi > 0, \qquad (1b)$$

$$\frac{\partial \theta}{\partial \eta} + H_2\,\theta = 0 \qquad \text{at}\, \eta = \eta_2, \qquad \xi > 0, \qquad (1c)$$

$$\theta = F(\eta) \quad \text{at}\, \xi = 0, \;\; \eta_1 \le \eta \le \eta_2, \qquad (1d)$$

$$\theta \to \theta_{fD} \quad \text{as}\, \xi \to \infty, \qquad (1e)$$

where the coefficients H_1 and H_2 do not vanish simultaneously and the boundary condition (1e) is as a consequence of the requirement that for large ξ the temperature distribution reduces to that of fully developed temperature profile, θ_{fD}. Various other quantities are defined as

$$\gamma(j) = 1 + \frac{\varepsilon_H(j)}{\alpha}, \quad j = x \,\text{or}\, y; \qquad \eta = \frac{y}{L};$$

$$\xi = \frac{x}{LPe}; \quad Pe = \frac{\beta \bar{u}_m L}{\alpha}; \quad u(\eta) = \beta \bar{u}_m \Delta(\eta),$$

$$H_i = \frac{h_i L}{k}, i = 1 \,\text{or}\, 2; \quad \theta = \frac{T}{T_0}; \quad G = \frac{L^2 g(\eta)}{T_0 k};$$

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$$s = \begin{cases} 0 & \text{for slab,} \\ 1 & \text{for cylinder.} \end{cases}$$

To solve the above problems we consider the eigenvalue problem appropriate for the case in which the axial conduction is neglected

$$\frac{1}{\eta^s}\frac{d}{d\eta}\left[\eta^s \gamma(\eta)\frac{d\psi}{d\eta}\right] + \lambda^2 \Delta(\eta)\psi(\eta) = 0 \quad \text{in } \eta_1 < \eta < \eta_2, \tag{2a}$$

$$-\frac{d\psi}{d\eta} + H_1\psi(\eta) = 0 \quad \text{at } \eta = \eta_1, \tag{2b}$$

$$\frac{d\psi}{d\eta} + H_2\psi(\eta) = 0 \quad \text{at } \eta = \eta_2. \tag{2c}$$

The eigenfunctions satisfy the following orthogonality condition

$$\int_{\eta_2}^{\eta_2} \eta^s\Delta(\eta)\psi(\lambda_m,\eta)\psi(\lambda_n,\eta)\,d\eta = \begin{cases} 0 & \text{for } m \neq n, \\ N(\lambda_m) & \text{for } m = n, \end{cases} \tag{3a}$$

where

$$N(\lambda_m) = \int_{\eta 1}^{\eta_2} \eta^s\Delta(\eta)\left[\psi(\lambda_m,\eta)\right]^2 d\eta. \tag{3b}$$

The integral transform pair with respect to the η variable is defined as

Transform:

$$\bar{\theta}(\xi,\lambda_m) = \int_{\eta_1}^{\eta_2} \eta'^s\Delta(\eta')\psi(\lambda_m,\eta')\,\theta(\xi,\eta')\,d\eta', \tag{4a}$$

Inversion:

$$\theta(\xi,\eta) = \sum_{m=1}^{\infty} \frac{1}{N(\lambda_m)}\psi(\lambda_m,\eta)\,\bar{\theta}(\xi,\lambda_m). \tag{4b}$$

We take integral transform of system (1) by the application of the transform (4a); namely, we operate on both sides of equation (1a) by the operator

$$\int_{\eta_1}^{\eta_2} \eta^s\psi(\lambda_m,\eta)\,d\eta,$$

and obtain

$$J_1 + \frac{1}{P^2 e}J_2 + J_3 = \frac{d\bar{\theta}(\xi,\lambda_m)}{d\xi}, \tag{5a}$$

where

$$J_1 = \int_{\eta_1}^{\eta_2} \psi(\lambda_m,\eta)\frac{\partial}{\partial\eta}\left[\eta^s\gamma(\eta)\frac{\partial\theta(\xi,\eta)}{\partial\eta}\right]d\eta, \tag{5b}$$

$$J_2 = \int_{\eta_1}^{\eta_2} \eta^s\psi(\lambda_m,\eta)\frac{\partial}{\partial\xi}\left[\gamma(\xi)\frac{\partial\theta}{\partial\xi}\right]d\eta, \tag{5c}$$

$$J_3 = \int_{\eta_1}^{\eta_2} \eta^s\psi(\lambda_m,\eta)\,G(\eta)\,d\eta. \tag{5d}$$

The term J_1 is evaluated by integrating it by parts twice and utilizing the eigenvalue problem (2) and the boundary conditions (1b) and (1c).

$$J_1 = -\lambda_m^2\,\bar{\theta}(\xi,\lambda_m). \tag{6a}$$

To evaluate the term J_2, the function $\theta(\eta,\xi)$ is replaced by its equivalent inversion formula (4b); We obtain

$$J_2 = \sum_{n=1}^{\infty}\frac{1}{N(\lambda_n)}\left[\int_{\eta_1}^{\eta_2}\eta^s\psi(\lambda_m,\eta)\psi(\lambda_n,\eta)\,d\eta\right]\frac{d}{d\xi}\left[\gamma(\xi)\frac{d\bar{\theta}(\xi,\lambda_n)}{d\xi}\right], \tag{6b}$$

where the summation index m of equation (4b) is replaced by n in order to distinguish it from the eventual inversion-formula summation-index m. The term J_3 is a known quantity because $G(\eta)$ is a specified function.

When equations (6a), (6b) and (5d) are introducted into equation (5a) we find

$$-\lambda_m^2\bar{\theta}_m(\xi) + \frac{1}{Pe^2}\sum_{n=1}^{\infty}\frac{(\psi_m,\psi_n)}{N_n}\frac{d}{d\xi}\left[\gamma(\xi)\frac{d\bar{\theta}_n(\xi)}{d\xi}\right] + (\psi_m,G) = \frac{d\bar{\theta}_m(\xi)}{d\xi}, \quad \text{for } \xi \to 0. \tag{7a}$$

The boundary conditions for this equation are obtained by taking the integral transform of the boundary conditions (1d) and (1e); that is

$$\bar{\theta}_m(\xi) = \bar{F}_m \quad \text{for } \xi = 0, \tag{7b}$$

$$\bar{\theta}_m(\xi) \to \bar{\theta}_{fD} \quad \text{as } \xi \to \infty, \tag{7c}$$

where the inner product of two functions is defined as

$$(\psi_m,G) \equiv \int_{\eta_1}^{\eta_2}\eta^s\psi_m(\eta)G(\eta)\,d\eta, \tag{8a}$$

$$(\psi_m,\psi_n) \equiv \int_{\eta_1}^{\eta_2}\eta^s\psi_m(\eta)\psi_n(\eta)\,d\eta, \tag{8b}$$

and the bar is used to denote the integral transform, i.e.

$$\bar{F}_m \equiv \int_{\eta_1}^{\eta_2}\eta^s\Delta(\eta)\psi_m(\eta)F(\eta)\,d\eta. \tag{8c}$$

Other abbreviations used include $\bar{\theta}_m(\xi) \equiv \bar{\theta}(\lambda_m,\xi)$, $\psi_m(\xi) \equiv \psi(\lambda_m,\eta)$, $N_n \equiv N(\lambda_m)$ and $m = 1, 2, 3, \ldots$.

Thus, the original problem (1) is transformed to the solution of an infinite set of coupled, second degree ordinary differential equations for the transforms $\bar{\theta}_m(\xi)$, $m = 1, 2, 3, \ldots$. Once $\bar{\theta}_m(\xi)$ are determined from the solution of this system, the temperature distribution $\theta(\xi,\eta)$ is immediately obtainable from the inversion formula (4b). However, it is not practical to solve an infinite set of coupled differential equations; but, simple approximations yielding highly accurate results are obtainable by considering only a finite number equations in the system as now described.

Lowest order analysis

The lowest order analysis is obtainable from this system by taking only one term in the series in equation (7a) by setting $m = n$. We find

$$-\lambda_m^2\bar{\theta}_m + \frac{1}{Pe^2}\frac{(\psi_m,\psi_m)}{N_m}\frac{d}{d\xi}\left[\gamma(\xi)\frac{d\bar{\theta}_m}{d\xi}\right] + (\psi_m,G) = \frac{d\bar{\theta}_m(\xi)}{d\xi}, \quad \text{for } \xi > 0, \tag{9a}$$

$$\bar{\theta}_m = \bar{F}_m \quad \text{for } \xi = 0, \tag{9b}$$

$$\bar{\theta}_m \to \bar{\theta}_{fD} \quad \text{for } \xi \to \infty, \tag{9c}$$

for $m = 1, 2, 3, \ldots$.

The system (9) provides a set of uncoupled, second-degree ordinary differential equations for the transforms $\bar{\theta}_m$, $m = 1, 2, 3, \ldots$. Once the transforms $\bar{\theta}$ are determined from the solution of this system, the temperature distribution $\theta(\xi,\eta)$ is immediately obtained by the inversion formula (4b).

The lowest order equations are also obtainable by other choices of the terms in the series; however, the choice given by equations (9) yields better results.

Higher order analysis

Higher order analysis is obtainable by considering a simultaneous solution of the first two or more of the differential equations; for the remainder of the functions $\bar{\theta}_m$, the system is reduced to a set of uncoupled second-degree differential equations as discussed above for the case of lowest order analysis.

LAMINAR FLOW INSIDE A CIRCULAR TUBE

To illustrate the application of the foregoing method of analysis and the accuracy of the lowest order solution, we consider laminar flow inside a circular tube with uniform wall temperature and a uniform inlet temperature at the origin of the axial coordinate. We allow for axial conduction in the downstream region of the origin. The mathematical formulation of the problem, in the dimensionless form, is given as

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right)+\frac{1}{Pe^2}\frac{\partial^2\theta}{\partial\xi^2} = (1-\eta^2)\frac{\partial\theta(\eta,\xi)}{\partial\xi}, \quad \text{in } 0<\eta<1,\ \xi>0, \tag{10a}$$

$$\frac{\partial\theta}{\partial\eta}=0 \qquad \text{at } \eta=0,\quad \xi>0, \tag{10b}$$

$$\theta(\eta,\xi)=0 \qquad \text{at } \eta=1,\quad \xi>0, \tag{10c}$$

$$\theta(\eta,\xi)=1 \qquad \text{at } \xi=0,\quad 0\le\eta\le1, \tag{10d}$$

$$\theta\to0 \qquad \text{at } \xi\to\infty, \tag{10e}$$

where

$$Pe=\frac{2\bar{u}_m R}{\alpha},\quad \xi=\frac{x}{RPe},\quad \eta=\frac{r}{R}.$$

The eigenvalue problem (2) takes the form

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}\psi}{\mathrm{d}\eta}\right)+\lambda^2(1-\eta^2)\psi(\eta)=0,\quad 0<\eta<1, \tag{11a}$$

$$\frac{\mathrm{d}\psi(\eta)}{\mathrm{d}\eta}=0 \qquad \text{at } \eta=0, \tag{11b}$$

$$\psi(\eta)=0 \qquad \text{at } \eta=1, \tag{11c}$$

which is the eigenvalue problem associated with the original Graetz problem.

The lowest order analysis for the transform $\bar{\theta}_m(\xi)$ of temperature is obtained from equations (9) as

$$-A_m\frac{\mathrm{d}^2\bar{\theta}_m(\xi)}{\mathrm{d}\xi^2}+\frac{\mathrm{d}\bar{\theta}_m(\xi)}{\mathrm{d}\xi}+\lambda_m^2\bar{\theta}_m(\xi)=0,\quad \xi>0, \tag{12a}$$

$$\bar{\theta}_m(\xi)=\bar{I}_m \qquad \text{for } \xi=0, \tag{12b}$$

$$\bar{\theta}_m(\xi)\to0 \qquad \text{as } \xi\to\infty, \tag{12c}$$

for $m = 1, 2, 3 \ldots$

where

$$A_m=\frac{(\psi_m,\psi_m)}{N_m Pe^2};\quad \bar{I}_m=\int_0^1\eta(1-\eta^2)\psi_m(\eta)\mathrm{d}\eta,$$

$$(\psi_m,\psi_m)=\int_0^1\eta[\psi_m(\eta)]^2\,\mathrm{d}\eta,$$

$$N_m=\int_0^1\eta(1-\eta^2)[\psi_m(\eta)]^2\,\mathrm{d}\eta.$$

The solution of equations (12) is

$$\bar{\theta}_m(\xi)=\bar{I}_m e^{-b_m\xi},\quad m=1,2,3,\ldots \tag{13}$$

where

$$b_m=\frac{-1+\sqrt{(1+4A_m\lambda_m^2)}}{2A_m}.$$

Then the temperature distribution $\theta(\eta,\xi)$ is determined as

$$\theta(\eta,\xi)=\sum_{m=1}^{\infty}\frac{1}{N_m}\psi_m(\eta)\bar{\theta}_m(\xi). \tag{14}$$

Once $\theta(\eta,\xi)$ is known, the average temperature $\theta_{av}(\xi)$ is calculated from

$$\theta_{av}(\xi)=4\int_0^1\eta(1-\eta^2)\theta(\eta,\xi)\,\mathrm{d}\eta, \tag{15}$$

and the Nusselt number from

$$Nu(\xi)=-2\frac{1}{\theta_{av}(\xi)}\left.\frac{\partial\theta(\eta,\xi)}{\partial\eta}\right|_{\eta=1}. \tag{16}$$

To illustrate the accuracy of the lowest order solution, we present in Table 1 a comparison of $\theta_{av}(\xi)$ and $Nu(\xi)$ obtained from the present analysis with those calculated by Singh [3]*. Clearly, the agreement is sufficiently close.

Table 1. A comparison of lowest order solution for θ_{av} and Nu with Singh's [3] results for $Pe = 50$†

	Lowest order solution		Singh's [3]	
ξ	θ_{av}	Nu	θ_{av}	Nu
0.005	0.896	7.905	0.91	8.079
0.025	0.718	4.697	0.73	4.759
0.05	0.580	4.027	0.59	4.034
0.10	0.396	3.716	0.40	3.715
0.25	0.132	3.657	0.134	3.66

† $Pe = 50$ in the present definition is equivalent to $Pe = 100$ in Singh's definition.

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* It was pointed out in [19] that an error was involved in the original analysis by Singh [2]; but that error does not invalidate his later results since he never actually used it in the subsequent developments.

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